MATH 224 B   Lecture 5

Computing the area of a surface

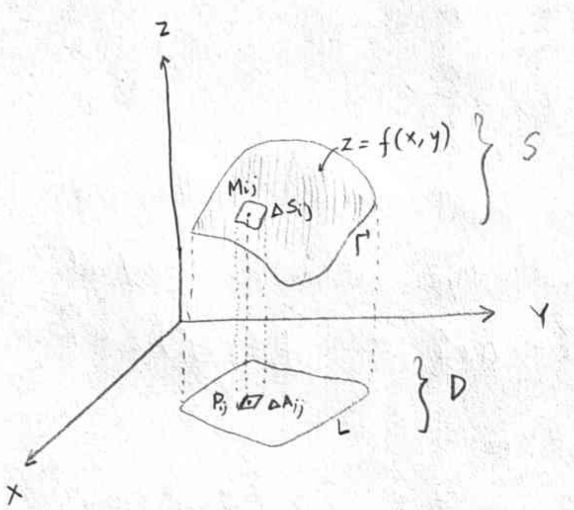


FIGURE-1

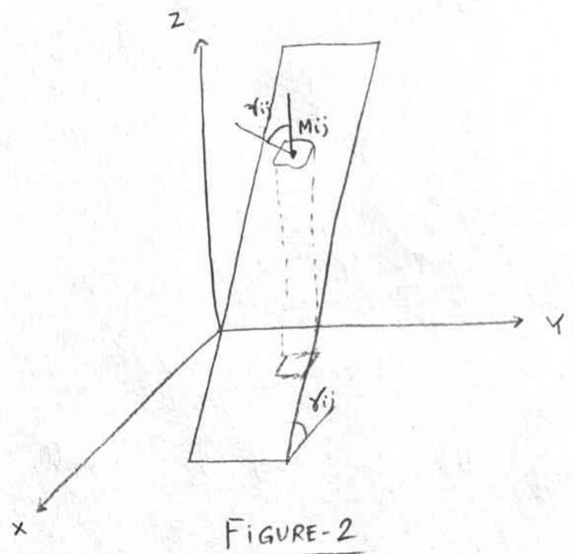


FIGURE-2

Consider FIGURE-1 and suppose the question is to find the area of the surface S $z = f(x,y)$ bounded by a curve Γ. Furthermore, suppose that $f(x,y)$ is a nice function (has continuous partial derivatives: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$ are continuous)

Suppose we project the surface down on the XY plane and let D denote the projected region. Let L be the projection of Γ.

Suppose we divide the region D into tiny subregions (say rectangles). In FIGURE-1, ΔA_{ij} denotes the area of one such rectangle.

Let $P_{ij}(x^*, y^*, 0)$ be a point inside the rectangle, and let $M_{ij}(x^*, y^*, f(x^*, y^*))$ corresponding point on the surface. Through M_{ij} we draw a tangent plane to the surface S as shown in FIGURE-2. Let γ_{ij} be the angle between the tangent plane and the XY plane. The equation of the tangent plane is given by:

$$z - f(x^*, y^*) = f_x(x^*, y^*)\,(x - x^*) + f_y(x^*, y^*)\,(y - y^*)$$

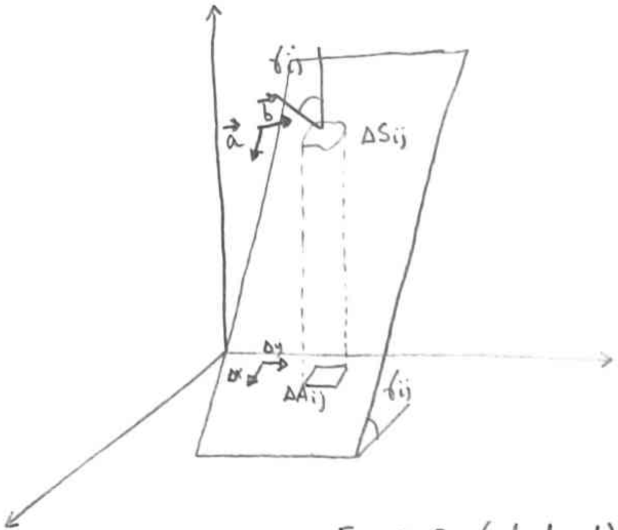


FIGURE 2 (reproduced)

$$\Delta A_{ij} = \left| \Delta x \,\hat{i} \times \Delta y \,\hat{j} \right| = \Delta x \,\Delta y$$

$$\Delta S_{ij} = \,?$$

If we project $\Delta x \,\hat{i}$ on the tangent plane, it would correspond to $\vec{a} = \Delta x \,\hat{i} + f_x(x^*, y^*)\,\Delta x \,\hat{k}$

Similarly, the vector $\Delta y \,\hat{j}$ would project to

$$\vec{b} = \Delta y \,\hat{j} + f_y(x^*, y^*)\,\Delta y \,\hat{k}.$$

Therefore

$$\Delta S_{ij} = \left| \vec{a} \times \vec{b} \right|$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(x^*, y^*)\,\Delta x \\ 0 & \Delta y & f_y(x^*, y^*)\,\Delta y \end{vmatrix}$$

$$= \left| -\Delta x \,\Delta y \, f_x(x^*, y^*)\,\hat{i} - \Delta x \,\Delta y \, f_y(x^*, y^*)\,\hat{j} + \Delta x \,\Delta y \,\hat{k} \right|$$

$$= \Delta x \,\Delta y \sqrt{1 + f_x^2(x^*, y^*) + f_y^2(x^*, y^*)}$$

[Alternatively, one can also use the fact that

$$\Delta S_{ij} \cos \gamma_{ij} = \Delta A_{ij}$$

where $$\cos(\gamma_{ij}) = \frac{1}{\sqrt{1 + f_x^2(x^*, y^*) + f_y^2(x^*, y^*)}}$$]

Therefore the total area

$$S \approx \sum_{i,j} \Delta S_{ij} = \sum \Delta A_{ij} \sqrt{1+f_x^2(x_i^*,y_j^*) + f_y^2(x_i^*,y_j^*)}$$

Therefore in the limit

$$S = \iint_D \sqrt{1+f_x^2(x,y) + f_y^2(x,y)} \; dA$$

$$= \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \; dx\, dy$$

This is the formula to compute area of the surface $z = f(x,y)$.

If the equation of the surface is given in the form
$x = \mu(y,z)$ or $y = \chi(x,z)$ then the corresponding formula for surface area becomes

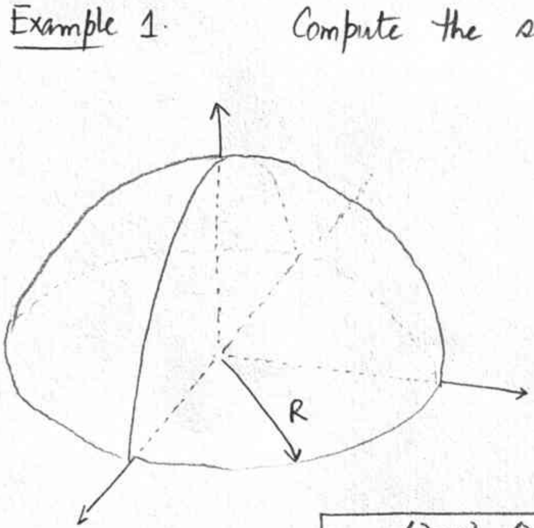
$$S' = \iint_{D'} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \; dy\, dz$$

and

$$S'' = \iint_{D''} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} \; dx\, dz .$$

where D' and D'' are the projected regions in the yz and xz plane of $\mu(y,z)$ and $\chi(x,z)$ respectively.

Example 1. Compute the surface area of the sphere $x^2+y^2+z^2=R^2$.



Lets compute the area of the hemisphere

$$z=\sqrt{R^2-x^2-y^2}$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2-x^2-y^2}} \quad ; \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2-x^2-y^2}}$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}} = \frac{R}{\sqrt{R^2-x^2-y^2}}$$

the region D in the xy plane is $x^2+y^2 \leq R^2$.

Therefore

$$\underbrace{\frac{1}{2}S}_{\substack{\text{Area of hemisphere} \\ \text{is } \frac{1}{2} \text{ Area of sphere}}} = \iint_D \frac{R}{\sqrt{R^2-x^2-y^2}}\, dA$$

(Polar)

$$= \int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-r^2}}\, r\,dr\, d\theta$$

let $u=R^2-r^2$, $du=-2r\,dr$

$$= \int_0^{2\pi}\int_{R^2}^{0} -\frac{R}{\sqrt{u}}\frac{du}{2}\, d\theta$$

$$= \int_0^{2\pi}\int_0^{R^2} \frac{R}{2}u^{-1/2}\,du\, d\theta$$

$$= \int_0^{2\pi} \frac{R}{2}\cdot\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_0^{R^2} d\theta = 2\pi R^2$$

$$\therefore \quad S = 2\cdot\left(2\pi R^2\right) = 4\pi R^2.$$