Lecture - 11 Chain Rule in Multivariable and Intro to Gradient.

Chain Rule

In single variable case Suppose g is a function of h which in turn is a function of x. In other words if for $g(h(x))$

then $$\frac{dg}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx}$$

An example: Let $g = h^2$ and $h(x) = \log_e x$

then $$g(h(x)) = (\log_e x)^2$$

and $$\frac{dg}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx} = 2h \cdot \frac{1}{x} = 2\log_e x \cdot \frac{1}{x}$$

In 2 variables we have :

Case 1 If $z = f(x,y)$ and $x = g(t)$, $y = h(t)$ then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

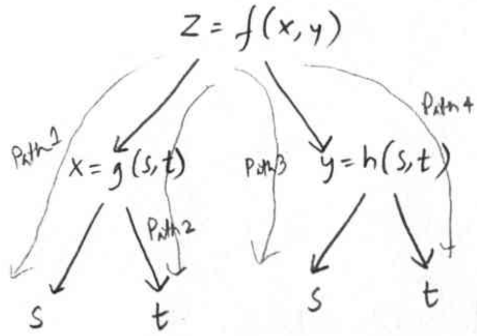
To Remember

$$\begin{tikzcd}
 & z = f(x,y) \arrow[dl] \arrow[dr] & \\
x \arrow[d, "\text{Path1}"] & & y \arrow[d, "\text{Path2}"'] \\
t & & t
\end{tikzcd}$$

Add all paths, where in each path take product of derivatives

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x} \cdot \frac{dx}{dt}}_{\text{Path1}} + \underbrace{\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}_{\text{Path2}}$$

Case 2 If $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$

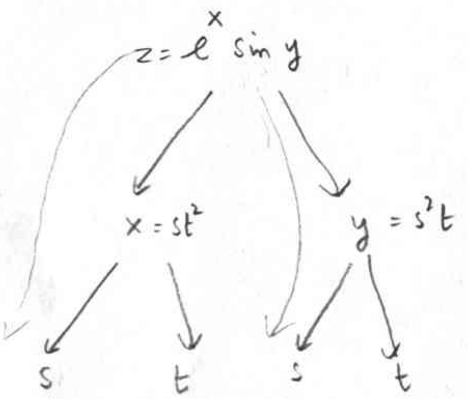


$$\frac{\partial z}{\partial s} = \underbrace{\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s}}_{\text{Path 1}} + \underbrace{\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}}_{\text{Path 3}}$$

$$\frac{\partial z}{\partial t} = \underbrace{\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t}}_{\text{Path 2}} + \underbrace{\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}}_{\text{Path 4}}$$

Example $z = e^x \sin y$ $\quad x = st^2$, $\quad y = s^2 t$.

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

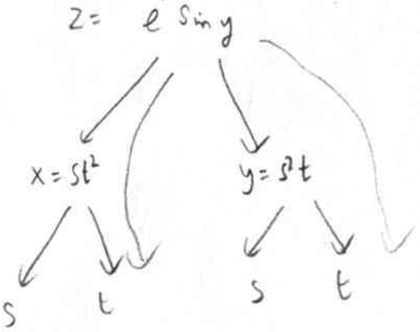
$$= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2s$$

$$= t^2 \cdot e^{st^2} \sin(s^2 t) + 2s \cdot e^{st^2} \cos(s^2 t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^x \sin y \cdot 2t + e^x \cos y \cdot s^2$$

$$= 2te^{st^2} \sin(s^2 t) + s^2 \cdot e^{st^2} \cos(s^2 t)$$



Chain Rule General Version

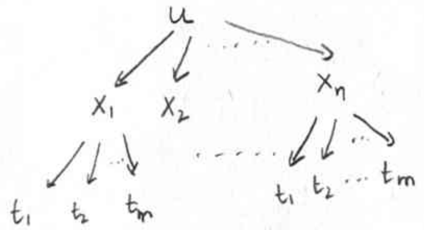
If $u(x_1, x_2, \dots, x_n)$ and

$x_1(t_1, \dots, t_m)$

$x_2(t_1, \dots, t_m)$

$\vdots$

$x_n(t_1, \dots, t_m)$

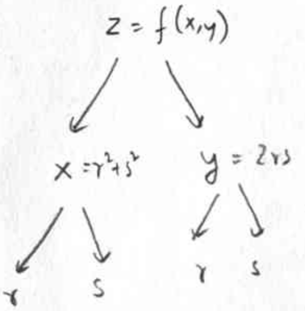


$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}}$$

Iterated Chain Rule

Suppose $z = f(x,y)$ and $x = r^2 + s^2$, $y = 2rs$

Find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$

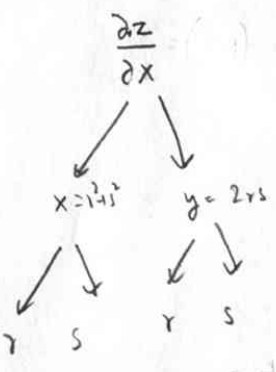


① $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \underbrace{\frac{\partial z}{\partial x}\cdot(2r) + \frac{\partial z}{\partial y}(2s)}$$

② $$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left[2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right]$$

(Product rule)
$$= 2r\,\underbrace{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)} + 2\frac{\partial z}{\partial x}\cdot 1 + 2s\,\underbrace{\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]} + 2\frac{\partial z}{\partial y}\left[\cancel{\frac{\partial s}{\partial r}}\right]^{0}$$

$$= 2r\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{dx}{dr} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial r}\right] + 2\frac{\partial z}{\partial x} + 2s\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}\right] +$$

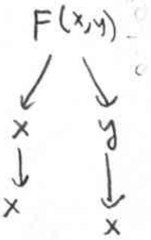


$$= 2r\left[\frac{\partial^2 z}{\partial x^2}\cdot 2r + \frac{\partial^2 z}{\partial x \partial y}\cdot 2s\right] + 2\frac{\partial z}{\partial x} + 2s\left[\frac{\partial^2 z}{\partial x \partial y}\cdot 2r + \frac{\partial^2 z}{\partial y^2}\cdot 2s\right]$$

$$= 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x}$$

Implicit Function, Chain Rule

① If $y=f(x)$ given implicitly as $F(x,y)=0$ and y is defined implicitly in terms of x then



$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

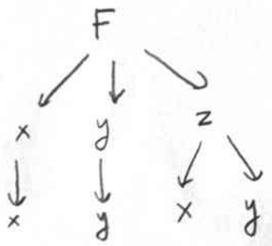
Example :

Find y' if $x^3+y^3 = 6xy$.

Write $F(x,y) = x^3+y^3-6xy=0$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2-6y}{3y^2-6x}$$

② If $z=f(x,y)$ is given implicity as $F(x,y,z)=0$



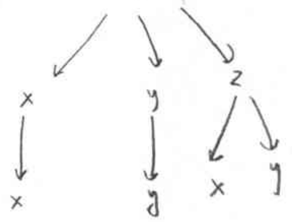
$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\cancel{\frac{\partial y}{\partial x}}^{\,0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x}\cdot\cancel{\frac{dx}{dy}}^{\,0} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dy} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}$$

This gives $$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Example $\quad \dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ when $\quad x^3+y^3+z^3+6xyz=1$

Let $\quad F(x,y,z) = x^3+y^3+z^3+6xyz-1=0$



$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

hence, $\quad \dfrac{\partial z}{\partial x} = -\dfrac{3x^2+6yz}{3z^2+6xy}$

Similarly $\quad \dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\dfrac{3y^2+6xz}{3z^2+6xy}$

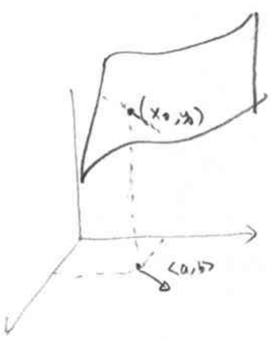
Introduction to gradient

- If $f(x,y)$ is a function of two variables then

  $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are the derivatives along $\underset{\langle 1,0\rangle}{x}$ and $\underset{\langle 0,1\rangle}{y}$ respectively.

Suppose I ask : What is the derivative along a direction $\vec{u}=\langle a, b\rangle$?

In other words ?: $\quad \displaystyle\lim_{h\to 0} \frac{f(x_0+ha,\ y_0+hb) - f(x_0,y_0)}{h}$



Ans: Its given by

$$D_{\vec{u}}\, f(x,y) = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b$$

where $\vec{u}=\langle a,b\rangle$ is a unit vector.

In order to write it more neatly we imagine the "derivative" being a vector and the directional derivative being the dot product:

$$D_u f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle$$

$$= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

The vector $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ is called the gradient denoted $\vec{\nabla} f$.

It allows us to treat the derivative as a vector function.

Example: Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ along $\vec{v} = 2\hat{i} + 5\hat{j}$.

Ans: Convert $\vec{v}$ to unit vector $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}$

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2xy^3, 3y^2x^2 - 4 \rangle$$

$$D_{\vec{u}} f(x,y) = \langle 2xy^3, 3y^2x^2 - 4 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

$$D_{\vec{u}} f(2,-1) = \langle -4, 8 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

$$= \frac{32}{\sqrt{29}}.$$